

Netted Binomial Matrices

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Abstract:

We prove that powers of 3-netted matrices (the entries satisfy a third-order recurrence $\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}$) preserve this property of nettedness, that is the entries of the e -th power satisfy $\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}$, where the coefficients are all instances of the same sequence, $x_{e+1} = (\beta + \delta)x_e - (\beta\delta + \alpha\gamma)x_{e-1}$. Also, we find a matrix $T_n(m)$ and a vector v , such that $T_n(m)^e \cdot v$ gives n consecutive entries of the general Fibonacci (Pell) sequence with parameter m . It generalizes the known property $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^e \cdot (1, 0)^t = (F_{e-1}, F_e)^t$. We close by giving a conjecture about the spectral properties of the binomial matrix we found.

1 Introduction

In [5], Peele and Stănică studied $n \times n$ matrices with the (i, j) entry the binomial coefficient $\binom{i-1}{j-1}$ (matrix L_n), respectively $\binom{i-1}{n-j}$ (matrix R_n) and derived many interesting results on powers of these matrices. L_n was easily subdued, but curiously enough, closed forms for entries of powers of R_n , say R_n^e , were not found. However, recurrences among various entries of R_n^e were proved and precise results on congruences modulo any prime p were found. They proved that the entries $a_{i,j}^{(e)}$ of the e -th power of R_n , satisfy

$$F_{e-1}a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1}a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)},$$

where F_e is the Fibonacci sequence, $F_{e+1} = F_e + F_{e-1}$, $F_0 = 0$, $F_1 = 1$, which was used to prove beautiful results on powers of these matrices modulo a prime p . As we shall see in our second section, this is not a singular phenomenon. In this paper we generalize the results of [5] for a class of matrices, containing R_n , where the entries satisfy a certain recurrence (we call these *netted* matrices). Moreover, we find a matrix with the property that any power multiplied by a fixed vector gives a tuple of consecutive terms of the Pell or Fibonacci numbers sequence. We also find the generating function for the entries of powers of these matrices. As applications we find four interesting identities for Fibonacci (or Pell) numbers. In the fifth section we provide a few results on the order of these matrices modulo a prime and in the last section we propose a conjecture on the spectral properties of $T_n(m)$.

2 Sequences Satisfying a Third Order Recurrence

Define a tableau with elements $a_{i,j}$, $i \geq 0$, $j \geq 0$, which satisfy (for $i \geq 1, j \geq 1$)

$$\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}, \quad (1)$$

with the boundary conditions

$$\beta a_{i,0} + \gamma a_{i+1,0} = 0, \forall 1 \leq i \leq n-1 \quad (2)$$

$$\delta a_{i+1,n+1} - \alpha a_{i,n+1} = 0, \forall 1 \leq i \leq n-1. \quad (3)$$

We remark that if the 0-th and $(n+1)$ -th column are made up of zeros, then the conditions (2) and (3) are fulfilled.

In our main result of this section we prove that (1) is preserved for higher powers of the $n \times n$ matrix $(a_{i,j})_{i=1\dots n, j=1\dots n}$. Precisely, we prove

Theorem 1. *The entries of the e -th power of the matrix $R = (a_{i,j})_{i=1\dots n, j=1\dots n}$ satisfy the recurrence*

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)},$$

where the sequences $\alpha_e, \beta_e, \gamma_e, \delta_e$ are all instances of the sequence x_e satisfying

$$x_{e+1} = (\beta + \delta)x_e - (\beta\delta + \alpha\gamma)x_{e-1},$$

with initial conditions $(\delta_1 = \delta; \delta_2 = \delta^2 - \alpha\gamma); (\alpha_1 = \alpha; \alpha_2 = \alpha(\delta + \beta)); (\beta_1 = \beta; \beta_2 = \beta^2 - \alpha\gamma)$ and $(\gamma_1 = \gamma; \gamma_2 = \gamma(\beta + \delta))$.

Proof. We prove by induction on e that there exists a relation among the entries of any 2×2 cells, namely

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}.$$

The above relation is certainly true for $n = 1$. We evaluate, for $i \geq 2$,

$$\begin{aligned}
\alpha \delta_{e-1} a_{i-1,j}^{(e)} &= \sum_{s=1}^n \alpha \delta_{e-1} a_{i-1,s} a_{s,j}^{(e-1)} \\
&= \sum_{s=1}^n \alpha a_{i-1,s} \left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} + \gamma_{e-1} a_{s,j-1}^{(e-1)} \right) \\
&= \sum_{s=1}^n (\delta a_{i,s} - \beta a_{i-1,s-1} - \gamma a_{i,s-1}) \left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} \right) \\
&\quad + \sum_{s=1}^n \alpha \gamma_{e-1} a_{i-1,s} a_{s,j-1}^{(e-1)} = \sum_{s=1}^n \delta a_{i,s} \left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} \right) \\
&\quad - \gamma \alpha_{e-1} a_{i,j}^{(e)} - \beta \beta_{e-1} a_{i-1,j-1}^{(e)} - \gamma \beta_{e-1} a_{i,j-1}^{(e)} - \beta \alpha_{e-1} a_{i-1,j}^{(e)} \\
&\quad + \alpha \gamma_{e-1} a_{i-1,j-1}^{(e)} - \gamma \alpha_{e-1} \left(a_{i,0} a_{0,j}^{(e-1)} - a_{i,n} a_{n,j}^{(e-1)} \right) \\
&\quad - \beta \beta_{e-1} \left(a_{i-1,0} a_{0,j-1}^{(e-1)} - a_{i-1,n} a_{n,j-1}^{(e-1)} \right) \\
&\quad - \gamma \beta_{e-1} \left(a_{i,0} a_{0,j-1}^{(e-1)} - a_{i,n} a_{n,j-1}^{(e-1)} \right) \\
&\quad - \beta \alpha_{e-1} \left(a_{i-1,0} a_{0,j}^{(e-1)} - a_{i-1,n} a_{n,j}^{(e-1)} \right).
\end{aligned}$$

Using the boundary conditions (2) and (3), we obtain, for $i \geq 2$,

$$\begin{aligned}
\alpha \delta_{e-1} a_{i-1,j}^{(e)} &= (\alpha \gamma_{e-1} - \beta \beta_{e-1}) a_{i-1,j-1}^{(e)} - \gamma \alpha_{e-1} a_{i,j}^{(e)} - \gamma \beta_{e-1} a_{i,j-1}^{(e)} \\
&\quad - \beta \alpha_{e-1} a_{i-1,j}^{(e)} + \sum_{s=1}^n \delta a_{i,s} \left(\delta_{e-1} a_{s,j}^{(e-1)} - \gamma_{e-1} a_{s,j-1}^{(e-1)} \right) \\
&\quad + \left(\alpha_{e-1} a_{n,j}^{(e-1)} + \beta_{e-1} a_{n,j-1}^{(e-1)} \right) (\beta a_{i-1,n} + \gamma a_{i,n}) \\
&\quad - \left(\alpha_{e-1} a_{0,j}^{(e-1)} + \beta_{e-1} a_{0,j-1}^{(e-1)} \right) (\beta a_{i-1,0} + \gamma a_{i,0}) \\
&= (\alpha \gamma_{e-1} - \beta \beta_{e-1}) a_{i-1,j-1}^{(e)} - \gamma \alpha_{e-1} a_{i,j}^{(e)} - \gamma \beta_{e-1} a_{i,j-1}^{(e)} \\
&\quad - \beta \alpha_{e-1} a_{i-1,j}^{(e)} + \delta \delta_{e-1} a_{i,j}^{(e)} - \delta \gamma_{e-1} a_{i,j-1}^{(e)} \\
&\quad + \left(\alpha_{e-1} a_{n,j}^{(e-1)} + \beta_{e-1} a_{n,j-1}^{(e-1)} \right) (\delta a_{i,n+1} - \alpha a_{i-1,n+1})
\end{aligned}$$

$$\begin{aligned}
&= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} + (\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)} \\
&\quad - (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)} - \beta\alpha_{e-1}a_{i-1,j}^{(e)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)} &= (\alpha\delta_{e-1} + \beta\alpha_{e-1})a_{i-1,j}^{(e)} + (\beta\beta_{e-1} - \alpha\gamma_{e-1})a_{i-1,j-1}^{(e)} \\
&\quad + (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)}.
\end{aligned}$$

Therefore, we obtain the system of sequences

$$\delta_e = \delta\delta_{e-1} - \gamma\alpha_{e-1} \quad (4)$$

$$\alpha_e = \alpha\delta_{e-1} + \beta\alpha_{e-1} \quad (5)$$

$$\beta_e = \beta\beta_{e-1} - \alpha\gamma_{e-1} \quad (6)$$

$$\gamma_e = \gamma\beta_{e-1} + \delta\gamma_{e-1}. \quad (7)$$

From (4) we get $\alpha_{e-1} = (\delta/\gamma)\delta_{e-1} - (1/\gamma)\delta_e$, which replaced in (5) gives the recurrence

$$\delta_{e+1} = (\beta + \gamma)\delta_e - (\beta\delta + \gamma\alpha)\delta_{e-1}.$$

Similarly,

$$\alpha_{e+1} = (\beta + \gamma)\alpha_e - (\beta\delta + \gamma\alpha)\alpha_{e-1}$$

$$\beta_{e+1} = (\beta + \gamma)\beta_e - (\beta\delta + \gamma\alpha)\beta_{e-1}$$

$$\gamma_{e+1} = (\beta + \gamma)\gamma_e - (\beta\delta + \gamma\alpha)\gamma_{e-1}.$$

The initial conditions are $(\delta_1 = \delta; \delta_2 = \delta^2 - \alpha\gamma), (\alpha_1 = \alpha; \alpha_2 = \alpha(\delta + \beta)), (\beta_1 = \beta; \beta_2 = \beta^2 - \alpha\gamma), (\gamma_1 = \gamma; \gamma_2 = \gamma(\beta + \delta))$. \square

Example 2. As examples of tableaux satisfying our conditions, we have $a_{i,j}^1 = \binom{i-1}{j-1}$ ($\delta = 1, \alpha = 1, \beta = 1, \gamma = 0$), $a_{i,j}^2 = \binom{i-1}{n-j}$ ($\delta = 0, \alpha = 1, \beta = 1, \gamma = -1$), $a_{i,j}^3 = \binom{n-i}{n-j}$

($\delta = 1, \alpha = 0, \beta = -1, \gamma = 1$). Other examples are given by the alternating matrices $(-1)^{i+j}a_{i,j}^k$ (or $(-1)^{i-1}a_{i,j}^k$ or $(-1)^{j-1}a_{i,j}^k$, etc.), $k = 1, 2, 3$. In the next section we present more examples.

3 Fibonacci and Pell Matrices

In this section we uncover a very interesting side of the previous section's results. A matrix of the form $M = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$ is called a *Fibonacci matrix*. It is known that if the sequence $U_{e+1} = mU_e + U_{e-1}$, $U_0 = 0$, $U_1 = 1$, then $M^e = \begin{pmatrix} U_{e-1} & U_e \\ U_e & U_{e+1} \end{pmatrix}$ and $M^e \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{e-1} \\ U_e \end{pmatrix}$. If m is an indeterminate, then U_e is called the *Fibonacci polynomial*. If $m = 2$, $U_e (= P_e)$ is the *Pell sequence*. The question that arises is whether there are higher dimensional square matrices $T_n(m)$ such that $T_n(m)^e \cdot v$ is a vector of n consecutive terms of the sequence U_n , for some vector v and any power e . We are able to answer positively the posed question for such a sequence. Let I_n be the identity matrix of dimension n and M^t be the *transpose* of a given matrix M .

First, we consider the Pell sequence: $P_{e+1} = 2P_e + P_{e-1}$, $P_0 = 0$, $P_1 = 1$. We prove that for each dimension there is a unique $n \times n$ matrix T_n with positive entries, constructed by bordering T_{n-1} and such that the entries satisfy $\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}$, with $\delta = 0, \alpha = 1, \beta = 2, \gamma = -1$. We also prove, using the previous section's result, that the entries of any power of this matrix satisfy a similar relationship, where the corresponding coefficients are all instances of the *Pell sequence*. Let $a_{i,j} = a_{i,j}^{(1)} = 2^{i+j-n-1} \binom{i-1}{n-j}$, $i, j \geq 0$.

Theorem 3. Let $v = ((-1)^n P_{n-1}, (-1)^{n-1} P_{n-2}, \dots, -P_0)^t$ and $T_n = (a_{i,j})_{1 \leq i,j \leq n}$. We

have $T_n^{e+1} \cdot v = (P_{(n-1)e}, P_{(n-1)e+1}, \dots, P_{(n-1)(e+1)})^t$ and

$$P_{e-1}a_{i,j}^{(e)} + P_e a_{i,j-1}^{(e)} = P_e a_{i-1,j}^{(e)} + P_{e+1} a_{i-1,j-1}^{(e)},$$

where $a_{i,j}^{(e)}$ are the entries of T_n^e and P_e is the Pell sequence. Moreover, T_n is unique with the property $a_{1,j} = 0, j < n, a_{i,n} = 2^{i-1}$ and $a_{i,j} = 2a_{i-1,j} + a_{i-1,j+1}$.

Proof. By induction on e we prove that $\sum_{j=1}^n (-1)^{n+1-j} a_{i,j}^{(e+1)} P_{n-j} = P_{(n-1)e+i-1}$, which will imply the first assertion. Assume $e = 0$. We need to show $\sum_{j=1}^n (-1)^{n+1-j} a_{i,j} P_{n-j} = P_{i-1}$, which will be proved by showing that the left hand side expression satisfies the Pell recurrence with the initial conditions of P_{i-1} . Denote by X_{i-1} the left hand side expression. First, $X_0 = \sum_{j=1}^n (-1)^{n+1-j} a_{1,j} P_{n-j} = (-1)^{n+1-n} a_{1,n} P_{n-n} = 0$. Now, $X_1 = \sum_{j=1}^n (-1)^{n+1-j} a_{2,j} P_{n-j} = (-1)^1 a_{2,n} P_0 + (-1)^2 a_{2,n} P_1 = 1$. Assume $1 \leq i \leq n-2$. Then

$$\begin{aligned} X_{i+1} &= \sum_{j=1}^n (-1)^{n+1-j} a_{i+2,j} P_{n-j} \\ &= \sum_{j=1}^n (-1)^{n+1-j} (2a_{i+1,j} + a_{i+1,j+1}) P_{n-j} \\ &= 2X_i + \sum_{j=1}^n (-1)^{n+1-j} a_{i+1,j+1} P_{n-j} \\ &= 2X_i + \sum_{j=1}^n (-1)^{n+1-j} a_{i+1,j+1} (2P_{n-j-1} + P_{n-j-2}) \\ &= 2X_i + 2 \sum_{j=1}^{n-1} (-1)^{n+1-j} a_{i+1,j+1} P_{n-(j+1)} \\ &\quad + \sum_{j=1}^{n-1} (-1)^{n+1-j} a_{i+1,j+1} P_{n-(j+1)-1} \\ &\stackrel{j+1=s}{=} 2X_i + 2 \sum_{s=2}^n (-1)^{n+2-s} a_{i+1,s} P_{n-s} + \sum_{s=2}^n (-1)^{n+2-s} a_{i+1,s} P_{n-s-1} \\ &= 2(-1)^n a_{i+1,1} P_{n-1} - Y_i + (-1)^n a_{i+1,1} P_{n-2} \\ &= (-1)^n a_{i+1,1} P_n - Y_i \stackrel{a_{i+1,1}=0 \text{ if } i \leq n-2}{=} -Y_i, \end{aligned}$$

where

$$\begin{aligned}
Y_i &\stackrel{def}{=} \sum_{s=1}^n (-1)^{n+1-s} a_{i+1,s} P_{n-s-1} \\
&= \sum_{s=1}^n (-1)^{n+1-s} (2a_{i,s} + a_{i,s+1}) P_{n-s-1} \\
&= 2 \sum_{s=1}^n (-1)^{n+1-s} a_{i,s} P_{n-s-1} + \sum_{s=1}^n (-1)^{n+1-s} a_{i,s+1} P_{n-s-1} \\
&= 2Y_{i-1} + \sum_{u=2}^n (-1)^{n+2-u} a_{i,u} P_{n-u} \\
&= 2Y_{i-1} - X_{i-2} + (-1)^n a_{i,1} P_{n-1} \stackrel{a_{i,1}=0 \text{ if } i \leq n-2}{=} 2Y_{i-1} - X_{i-2}
\end{aligned}$$

Using $Y_i = -X_{i+1}$ in the previous recurrence we get

$$X_{i+1} = 2X_i + X_{i-1},$$

relation satisfied by the Pell sequence. Since X_{i-1} has the same initial conditions as P_{i-1}

we have $\sum_{j=1}^n (-1)^{n+1-j} a_{i,j} P_{n-j} = P_{i-1}$.

The first step of induction is proven. Now,

$$\begin{aligned}
&\sum_{j=1}^n (-1)^{n+1-j} a_{i,j}^{(e+1)} P_{n-j} = \sum_{j=1}^n (-1)^{n+1-j} \sum_{k=1}^n a_{i,k} a_{k,j}^{(e+1)} P_{n-j} \\
&= \sum_{k=1}^n a_{i,k} \sum_{j=1}^n (-1)^{n+1-j} a_{k,j}^{(e)} P_{n-j} \\
&= \sum_{j=1}^n a_{i,k} P_{(n-1)(e-1)+k-1}.
\end{aligned}$$

We shall prove that the matrix T acts as an index-translation on the Pell sequence, namely

$$\sum_{k=1}^n a_{i,k} P_{t+k} = P_{t+n+i-1}, \quad t \geq -1.$$

If this is so, then by taking $t = (n-1)(e-1) - 1$, the step of induction will be done. Let

$W_i = \sum_{k=1}^n a_{i,k} P_{t+k}$ (t is assumed fixed). First, $W_1 = \sum_{k=1}^n a_{1,k} P_{t+k} = a_{1,n} P_{t+n} = P_{t+n}$.

Then, $W_2 = \sum_{k=1}^n a_{2,k} P_{t+k} = a_{2,n-1} P_{t+n-1} + a_{2,n} P_{t+n} = P_{t+n-1} + 2P_{t+n} = P_{t+n+1}$. Now, for $1 \leq i \leq n-1$,

$$\begin{aligned}
W_{i+1} &= \sum_{k=1}^n a_{i+1,k} P_{t+k} = \sum_{k=1}^n (2a_{i,k} + a_{i,k+1}) P_{t+k} \\
&= 2W_i + \sum_{k=1}^n a_{i,k+1} P_{t+k} = 2W_i + \sum_{k=1}^{n-1} a_{i,k+1} (P_{t+k+2} - 2P_{t+k+1}) \\
&\stackrel{u=k+1}{=} 2W_i + \sum_{u=2}^n a_{i,u} P_{t+u+1} - 2 \sum_{u=2}^n a_{i,u} P_{t+u} \\
&= V_i - a_{i,1} P_{t+2} + 2a_{i,1} P_{t+1} \stackrel{a_{i,1}=0 \text{ if } i \leq n-1}{=} V_i,
\end{aligned}$$

where

$$\begin{aligned}
V_i &= \sum_{u=1}^n a_{i,u} P_{t+u+1} = \sum_{u=1}^n (2a_{i-1,u} + a_{i-1,u+1}) P_{t+u+1} \\
&\stackrel{a_{i+1,n+1}=0}{=} 2V_{i-1} + \sum_{u=1}^{n-1} a_{i-1,u+1} P_{t+u+1} \stackrel{u+1=s}{=} 2V_{i-1} + \sum_{u=2}^n a_{i-1,s} P_{t+s} \\
&= 2V_{i-1} + X_{i-1} - a_{i-1,1} P_{t+1} \stackrel{a_{i-1,1}=0 \text{ if } i \leq n-1}{=} 2V_{i-1} + W_{i-1}.
\end{aligned}$$

Using $V_i = W_{i+1}$ in the previous recurrence, we get $W_{i+1} = 2W_i + W_{i-1}$. Therefore,

$W_i = P_{t+n+i-1}$, since $W_1 = P_{t+n}$, $W_2 = P_{t+n+1}$.

Using Theorem 1, with $\delta = 0, \alpha = 1, \beta = 2, \gamma = -1$, we get the recurrence between the entries of the higher power of T_n , namely $P_{e-1} a_{i,j}^{(e)} + P_e a_{i,j-1}^{(e)} = P_e a_{i-1,j}^{(e)} + P_{e+1} a_{i-1,j-1}^{(e)}$.

The fact that T_n is the unique matrix with the given properties follows easily observing that such a matrix could be defined inductively as follows: let $T_1 = 1$. Assume $T_{n-1} = (a_{i,j})_{i,j=1,2,\dots,n-1}$ and construct T_n by bordering T_{n-1} with the first column and the last row (left and bottom). The first column is $(0, 0, \dots, 0, 1)^t$ and the last row is given by: $a_{n,n} = 2^{n-1}$ and $a_{n,j} = 2a_{n-1,j} + a_{n-1,j+1}$. \square

Let $a_{i,j} = a_{i,j}^{(1)} = m^{i+j-n-1} \binom{i-1}{n-j}$. Similarly, we can show (we omit the proof)

Theorem 4. Let $w = ((-1)^n U_{n-1}, (-1)^{n-1} U_{n-2}, \dots, -U_0)^t$ and $T_n(m) = (a_{i,j})_{i,j}$. Then $T_n(m)^{e+1} \cdot w = (U_{(n-1)e}, U_{(n-1)e+1}, \dots, U_{(n-1)(e+1)})^t$ and

$$U_{e-1} a_{i,j}^{(e)} + U_e a_{i,j-1}^{(e)} = U_e a_{i-1,j}^{(e)} + U_{e+1} a_{i-1,j-1}^{(e)}, \quad (8)$$

where $a_{i,j}^{(e)}$ are the entries of $T_n(m)^e$ and U_e is the sequence satisfying $U_{e+1} = mU_e + U_{e-1}$, $U_0 = 0$, $U_1 = 1$. Moreover, $T_n(m)$ is unique with the property $a_{1,j} = 0, j < n, a_{i,n} = m^{i-1}$ and $a_{i,j} = ma_{i-1,j} + a_{i-1,j+1}$.

Definition 5. We call such a matrix $T_n(m)$ a generalized Fibonacci matrix of dimension n and parameter m . If $m = 2$, $T_n(2) = T_n$ is the Pell matrix.

Example 6. We give here the first few powers of $T_3(m)$,

$$T_3(m) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & m \\ 1 & 2m & m^2 \end{pmatrix}, \quad T_3(m)^2 = \begin{pmatrix} 1 & 2m & m^2 \\ m & 1+2m^2 & m+m^3 \\ m^2 & 2(m+m^3) & (1+m^2)^2 \end{pmatrix},$$

$$T_3(m)^3 = \begin{pmatrix} m^2 & 2(m+m^3) & (1+m^2)^2 \\ m+m^3 & 1+4m^2+2m^4 & m(2+3m^2+m^4) \\ (1+m^2)^2 & 2m(2+3m^2+m^4) & m^2(2+m^2)^2 \end{pmatrix}$$

By taking some particular cases of our previous results we get some very interesting binomial sums. For instance,

Corollary 7. We have

1. $\sum_{j=1}^n (-1)^{n+1-j} m^{i+j-n-1} \binom{i-1}{n-j} U_{n-j} = U_{i-1}.$
2. $\sum_{j=1}^n \sum_{k=1}^n (-1)^{n+1-j} m^{i+j+2k-2n-2} \binom{i-1}{n-k} \binom{k-1}{n-j} U_{n-j} = U_{n+i-2}.$
3. $\sum_{j=1}^n U_{l-1}^{n-j} U_l^{j-1} U_{(n-1)p+j-1} \binom{n-1}{j-1} = U_{(n-1)(l+p)}, \text{ for any } l, p.$
4. $\sum_{j=1}^n U_{(n-1)p+j-1} U_{l-1}^{n-j-1} U_l^{j-2} \left[U_l^2 \binom{n-1}{j-1} + (-1)^l \binom{n-2}{j-2} \right] = U_{(n-1)(l+p)+1}, \text{ for any } l, p.$

Proof. Using Theorem 4, with $e = 1, 2$, we obtain the first two identities. Now, with the help of Theorem 4 and the trivial identity $T_n(m)^{l+p} = T_n(m)^l T_n(m)^p$, we get

$$\begin{aligned} \left(T_n(m)^l T_n(m)^p \right) \cdot v &= T_n(m)^l \cdot \left(U_{(n-1)p}, U_{(n-1)p+1}, \dots, U_{(n-1)(p+1)} \right)^t \\ &= \left(U_{(n-1)(l+p)}, U_{(n-1)(l+p)+1}, \dots, U_{(n-1)(l+p+1)} \right). \end{aligned}$$

Since $a_{1,j}^{(l)} = U_{l-1}^{n-j} U_l^{j-1} \binom{n-1}{j-1}$, we obtain the third identity. Using $a_{2,j}^{(l)} = U_{l-1}^{n-j-1} U_l^j \binom{n-2}{j-1} + U_{l-1}^{n-j} U_e^{j-2} U_{l+1} \binom{n-2}{j-2}$ and Cassini's identity (see [3], p. 292) (usually given for the Fibonacci numbers, but certainly true for the sequence U_l , as well, as the reader can check easily, by induction) $U_{l-1} U_{l+1} - U_l^2 = (-1)^l$, we get the fourth identity. \square

Remark 8. *In general, we have*

$$\sum_{j=1}^n U_{(n-1)p+j-1} a_{i,j}^{(l)} = U_{(n-1)(l+p)+i-1}$$

for any i, l, p .

In general, finding closed forms for the entries of powers of $T_n(m)$ seems to be a very difficult matter. We can derive (after some work) simple formulas for the entries of the second row and column of $T_n(m)^e$.

Proposition 9. *We have $a_{2,j}^{(e)} = U_{e-1}^{n-j-1} U_e^j \binom{n-2}{j-1} + U_{e-1}^{n-j} U_e^{j-2} U_{e+1} \binom{n-2}{j-2}$, and $a_{i,2}^{(e)} = (n-i) U_{e-1}^{n-i} U_e^{i-1} + (i-1) U_{e-1}^{n-i+1} U_e^{i-2} U_{e+1}$.*

Remark 10. *Since $a_{j,n}^{(e-1)} = a_{j,1}^{(e)}, a_{n,j}^{(e-1)} = a_{1,j}^{(e)}$, we get closed forms for the last row and column of $T_n(m)^e$, as well.*

4 Some Generating Functions and an Inverse

Although we cannot find simple closed forms for *all* entries of $T_n(m)^e$, we prove

Theorem 11. *The generating function for $a_{i,j}^{(e)}$ is*

$$B_n^{(e)}(x, y) = \frac{(U_{e-1} + U_e y)^n}{U_{e-1} + U_e y - x(U_e + U_{e+1}y)}$$

Proof. Multiplying the recurrence (8) by $x^{i-1}y^{j-1}$ and summing for $i, j \geq 2$, we get

$$\begin{aligned} U_{e-1} \sum_{i,j \geq 2} a_{i,j}^{(e)} x^{i-1} y^{j-1} + U_e y \sum_{i,j \geq 2} a_{i,j-1}^{(e)} x^{i-1} y^{j-2} = \\ U_e x \sum_{i,j \geq 2} a_{i-1,j}^{(e)} x^{i-2} y^{j-1} + U_{e+1} x y \sum_{i,j \geq 2} a_{i-1,j-1}^{(e)} x^{i-2} y^{j-2} \end{aligned}$$

Thus,

$$\begin{aligned} U_{e-1} \left(B_n^{(e)}(x, y) - \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} - \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} + a_{1,1}^{(e)} \right) \\ + U_e y \left(B_n^{(e)}(x, y) - \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} \right) = \\ U_e x \left(B_n^{(e)}(x, y) - \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} \right) + U_{e+1} x y B_n^{(e)}(x, y). \end{aligned}$$

Solving for $B_n^{(e)}(x, y)$, we get

$$\begin{aligned} B_n^{(e)}(x, y) (U_{e-1} + U_e y - x(U_e + U_{e+1}y)) = \\ (U_{e-1} - U_e x) \sum_{i \geq 1} a_{i,1}^{(e)} x^{i-1} + (U_{e-1} + U_e y) \sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} - U_{e-1} a_{1,1}^{(e)}. \end{aligned} \tag{9}$$

We need to find $a_{i,1}^{(e)}$ and $a_{1,j}^{(e)}$. We prove

$$\begin{aligned} a_{1,j}^{(e)} &= \binom{n-1}{j-1} U_{e-1}^{n-j} U_e^{j-1} \\ a_{i,1}^{(e)} &= U_{e-1}^{n-i} U_e^{i-1}. \end{aligned} \tag{10}$$

There is no difficulty to show the relations for $e = 1, 2$. Assume $e \geq 3$. First we deal with the elements in the first row,

$$\begin{aligned}
a_{1,j}^{(e+1)} &= \sum_{s=1}^n a_{1,s}^{(e)} a_{s,j} = \sum_{s=1}^n U_{e-1}^{n-s} U_e^{s-1} m^{s+j-n-1} \binom{n-1}{s-1} \binom{s-1}{n-j} \\
&= \sum_{s=1}^n U_{e-1}^{n-s} U_e^{s-1} m^{s+j-n-1} \binom{n-1}{j-1} \binom{j-1}{n-s} \\
&= m^{j-1} U_e^{n-1} \binom{n-1}{j-1} \sum_{s=1}^n \left(\frac{U_{e-1}}{m U_e} \right)^{n-s} \binom{j-1}{n-s} \\
&= m^{j-1} U_e^{n-1} \binom{n-1}{j-1} \left(1 + \frac{U_{e-1}}{m U_e} \right)^{j-1} = U_e^{n-j} U_{e+1}^{j-1} \binom{n-1}{j-1}.
\end{aligned}$$

Now we prove the result for the elements in the first column.

$$\begin{aligned}
a_{i,1}^{(e+1)} &= \sum_{s=1}^n a_{i,s}^{(e)} a_{s,1} = \sum_{s=1}^n m^{i+s-n-1} \binom{i-1}{n-s} U_{e-1}^{n-s} U_e^{s-1} \\
&= m^{i-1} U_e^{n-1} \sum_{s=1}^n \left(\frac{U_{e-1}}{m U_e} \right)^{n-s} \binom{i-1}{n-s} \\
&= m^{i-1} U_e^{n-1} \left(1 + \frac{U_{e-1}}{m U_e} \right)^{i-1} = U_e^{n-i} U_{e+1}^{i-1}.
\end{aligned}$$

Using (10), we get

$$\begin{aligned}
\sum_{j \geq 1} a_{1,j}^{(e)} y^{j-1} &= \sum_{j \geq 1} \binom{n-1}{j-1} U_{e-1}^{n-j} U_e^{j-1} y^{j-1} \\
&= \sum_{s \geq 0} \binom{n-1}{s} U_{e-1}^{(n-1)-s} (y U_e)^s \\
&= (U_{e-1} + y U_e)^{n-1}
\end{aligned}$$

Using (9) and the fact that $(U_{e-1} - U_e x) \sum_{i \geq 1} U_{e-1}^{n-i} U_e^{i-1} x^{i-1} = U_{e-1}^n$ and $U_{e-1} a_{1,1}^{(e)} = U_{e-1}^n$, we deduce the result. \square

The inverse of $T_n(m)$ is not difficult to find. We have

Theorem 12. The inverse of $T_n(m) = \left(m^{i+j-n-1} \binom{i-1}{n-j}\right)_{i,j}$ is

$$T_n(m)^{-1} = \left((-1)^{n+i+j+1} m^{n+1-i-j} \binom{n-i}{j-1}\right)_{i,j}.$$

Proof. The (i, j) entry in $A^{-1}A$ is

$$\begin{aligned} & \sum_{s=1}^n (-1)^{n+i+s+1} m^{n+1-i-s} \binom{n-i}{s-1} m^{s+j-n-1} \binom{s-1}{n-j} \\ &= m^{j-i} \sum_{s=1}^n (-1)^{n+i+s+1} \frac{(n-i)!}{(s-1)!(n-i-s+1)!} \frac{(s-1)!}{(n-j)!(s+j-n-1)!} \\ &= m^{j-i} \sum_{s=1}^n (-1)^{n+i+s+1} \frac{(n-i)!}{(n-j)!(j-i)!} \frac{(j-i)!}{(n-i-s+1)!(s+j-n-1)!} \\ &= m^{j-i} \binom{n-i}{n-j} \sum_{s=1}^n (-1)^{n+i+s+1} \binom{j-i}{n-i-s+1} \\ &= m^{j-i} \binom{n-i}{n-j} \sum_{k=0}^{j-i} (-1)^k \binom{j-i}{k}, \end{aligned}$$

which is 0, unless $i = j$, in which case it is 1. □

5 Powers of $T_n(m)$ modulo p

Let $m \in \mathbf{Z}$. Using the recurrence among the entries of $T_n(m)$, and reasoning as in [5], we prove the following

Theorem 13. If e is the least integer (entry point) such that $U_e \equiv 0 \pmod{p}$, then

$$T_{2k}(m)^e \equiv (-1)^{(k+1)e} U_{e-1} I_{2k} \pmod{p}$$

$$T_{2k+1}(m)^e \equiv (-1)^{ke} I_{2k+1} \pmod{p}.$$

Moreover, $T_n(m)^{4e} \equiv I_n \pmod{p}$. Furthermore, considering the parity of e , we have

$$T_n(m)^{2e} \equiv I_n \pmod{p} \text{ if } e \text{ even}$$

and if e odd

$$T_n(m)^{2e} \equiv r^{n-1} I_n \pmod{p} \text{ if } e \equiv 3 \pmod{4}$$

$$T_n(m)^{2e} \equiv (-r)^{n-1} I_n \pmod{p} \text{ if } e \equiv 1 \pmod{4},$$

where $r \equiv \frac{U_{(e+1)/2}}{U_{(e-1)/2}} \pmod{p}$, so $r^2 \equiv -1 \pmod{p}$.

Proof. Using (8), if $U_e \equiv 0 \pmod{p}$, then

$$U_{e-1}a_{i,j}^{(e)} \equiv U_{e+1}a_{i-1,j-1}^{(e)}.$$

Since p divides neither U_{e-1} nor U_{e+1} (otherwise it would divide $U_1 = 1$), we get

$$a_{i,j} \equiv 0 \pmod{p}, \text{ if } i \neq j,$$

$$a_{i,i}^{(e)} \equiv a_{i-1,i-1}^{(e)} \equiv \cdots \equiv a_{1,1}^{(e)} \equiv U_{e-1}^{n-1} \pmod{p}.$$

Therefore

$$T_n(m)^e \equiv U_{e-1}^{n-1} I_n \pmod{p}.$$

Using Cassini's identity $U_{l-1}U_{l+1} - U_l^2 = (-1)^l$, for $l = e$, we get, if $n = 2k$,

$$U_{e-1}^{n-1} = U_{e-1}^{2k-1} \equiv (U_{e-1}^2)^k U_{e-1}^{-1} \equiv (-1)^{ke} U_{e-1}^{-1} \equiv (-1)^{(k+1)e} U_{e-1} \pmod{p},$$

since $U_{e-1}^2 \equiv U_{e+1}^2 \equiv (-1)^e \pmod{p}$. If $n = 2k + 1$, then

$$U_{e-1}^{n-1} = U_{e-1}^{2k} \equiv (U_{e-1}^2)^k \equiv (-1)^{ke} \pmod{p}.$$

The previous two congruences replaced in $T_n(m)^e \equiv U_{e-1}^{n-1} I_n \pmod{p}$, proves the first claim.

Lemma 3.4 of [4] implies

$$U_{e-1} \equiv (-1)^{\frac{e-2}{2}} \text{ if } e \text{ even}$$

$$U_{e-1} \equiv r(-1)^{\frac{e-3}{2}}, r^2 \equiv -1 \pmod{p}, \text{ if } e \text{ odd}.$$

The residue r in the previous relation is just $r \equiv \frac{U_{(e+1)/2}}{U_{(e-1)/2}} \pmod{p}$. Thus, if e even, then

$$U_{e-1}^2 \equiv 1 \pmod{p}, \text{ so}$$

$$T_n(m)^{2e} \equiv I_n \pmod{p},$$

for any n . The remaining cases are similar. □

Similarly, we can prove

- Theorem 14.** 1. If $p \mid U_{p-1}$, then $T_n(m)^{p-1} \equiv I_n \pmod{p}$.
 2. If $p \mid U_{p+1}$, then $T_{2k+1}(m)^{p+1} \equiv I_{2k+1} \pmod{p}$ and $T_{2k}(m)^{p+1} \equiv -I_{2k} \pmod{p}$.

A consequence of Theorem 1 of [1] is

Lemma 15. For a prime p which divides $f(x) = x^2 - mx - 1$ for an integer x , the sequence $\{U_e\}_e$ has a period $p - 1 \pmod{p}$, provided p is not a divisor of $D = m^2 + 4$.

Our final result is

Theorem 16. Let p be a prime divisor of $x^2 - mx - 1$, for some integer x and $\gcd(p, m^2 + 4) = 1$. Then, $T_n(m)^{p-1} \equiv I_n \pmod{p}$.

Proof. Straightforward, using Lemma 15 and Theorem 4 or Theorem 13. □

6 Further Research

We observed that netted matrices defined using second/third-order recurrences (we call these 2 or 3-netted matrices) preserve a third-order recurrence among the entries of their powers. The natural question arising is: *what is the degree of the recurrence (if it exists - we conjecture that it does) for higher powers of a 4-netted, 5-netted, etc., matrix?*

The spectral properties of $T_n(m)$ is another topic of future research. Let $U_e = U_e(m)$ be the general Pell or Fibonacci sequence. It is known that $U_e = \frac{\alpha^e - \beta^e}{\alpha - \beta}$, where $\alpha = \frac{m + \sqrt{m^2 + 4}}{2}$, $\beta = \frac{m - \sqrt{m^2 + 4}}{2}$. We associate the general Lucas sequence V_e satisfying the same recurrence as U_e , with initial conditions $V_0 = 2$, $V_1 = m$. Thus $V_e = \alpha^e + \beta^e$. We conjecture

Conjecture 17. *The characteristic polynomial $p_n(x)$ of $T_n(m)$ is*

$$p_1(x) = 1 - x; \quad p_{4k+1} = (1 + V_{4k-2}x + x^2)(1 - V_{4k}x + x^2)p_{4k-3}(x)$$

$$p_3(x) = -(1 + x)(1 - V_2x + x^2); \quad p_{4k+3} = (1 + V_{4k}x + x^2)(1 - V_{4k+2}x + x^2)p_{4k-1}(x)$$

$$p_2(x) = -1 - V_1x + x^2; \quad p_{4k+2} = (-1 + V_{4k-1}x + x^2)(-1 - V_{4k+1}x + x^2)p_{4k-2}(x)$$

$$p_0(x) \stackrel{def}{=} 1; \quad p_{4k+4} = (-1 + V_{4k+1}x + x^2)(-1 - V_{4k+3}x + x^2)p_{4k}(x).$$

We checked the conjecture up to dimension 100×100 .

References

- [1] S. Ando, *On the Period of Sequences Modulo a Prime Satisfying a Second Order Recurrence*, Applications of Fibonacci Numbers, Vol. 7, 1998, pp. 17-22.
- [2] L.E. Dickson, *History of the Theory of Numbers*, Vol. 1, Ch. XVII, Chelsea Publishing Co., 1971.
- [3] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics*, Adison-Wesley Publishing Company, 1989.
- [4] H.-C. Li, *On Second-Order Linear Recurrence Sequences: Wall and Wyler Revisited*, Fibonacci Quarterly, Nov. 1999, pp. 342-349.
- [5] R. Peele, P. Stănică, *Matrix Powers of Column-Justified Pascal Triangles and Fibonacci Sequences*, to appear (available at <http://sciences.aum.edu/~stanpan>).
- [6] M. Petkovsek, H. Wilf, D. Zeilberger, *A = B*, A.K. Peters, Ltd., Wellesley, Massachusetts, 1997.